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Computation of the Largest Lyapunov Exponent by the Generalized Cell Mapping

Myun C. Kim[1] **and C. S. Hsu**[1]

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The method of cell mappings has been developed as an efficient tool for the global study of dynamical systems. One of them, the generalized cell mapping (GCM), describes the behavior of a system in a probabilistic sense, and is essentially a Markov chain analysis of dynamical systems. Since the largest Lyapunov exponent is widely used to characterize attractors of dynamical systems, we propose an algorithm for that quantity by the GCM. This allows us to examine the persistent groups of the GCM in terms of their Lyapunov exponent, thereby connecting them with their counterparts in point mapping systems.

KEY WORDS: Strange attractors; Lyapunov exponents; cell-to-cell mapping; generalized cell mapping; nonlinear dynamical systems; Hénon–Pomeau map; forced Duffing system.

1. INTRODUCTION

During the past two decades, the theory of dynamical systems has advanced rapidly. Among the many exciting developments, the phenomena of strange attractors and chaotic motion have drawn a great deal of attention from researchers in many different scientific disciplines (see references in Refs. 1 and 2).

In the meanwhile the method of cell mapping has been developed and has proven to be a promising tool for the global study of dynamical systems. The state space is discretized into a large collection of cells, and a dynamical system is described in terms of a cell-to-cell mapping. Two kinds of cell mapping have been and are being further developed: simple cell mapping[(3,4)] and generalized cell mapping.[(5,6)] The generalized cell mapping (GCM) describes the behavior of a system in a probabilistic sense, with

[1] Department of Mechanical Engineering, University of California, Berkeley, California 94720.

the theory of the Markov chain as the basic tool of analysis. The GCM has been applied successfully to find the domain of attraction for the attractors in nonlinear dynamical systems and to study the statistical properties of strange attractors[7] as well as random vibration problems.[8] It also served as the basis in the evaluation of the metric entropy for a certain class of maps of the interval.[9,10]

In this paper attractors of dynamical systems are studied with generalized cell mapping. They are represented as persistent groups in GCM. A computer algorithm is developed to compute the largest Lyapunov exponent by the method of GCM. Examples are drawn from two-dimensional point mapping systems and differential equations; the largest Lyapunov exponents of their attractors are computed by the proposed algorithm. The results indicate the effectiveness of the proposed algorithm in evaluating the largest Lyapunov exponent.

2. ATTRACTORS OF DYNAMICAL SYSTEMS

Consider a (continuous) dynamical system governed by the differential equations

$$\frac{dx}{dt} = F(x, t), \qquad x \in U \subset R^N, \quad t \in I \subset R \tag{1}$$

where x is an N-vector and $F\colon U \times I \to R^N$ is a smooth function. The vector field F generates a flow $\varphi(x, t)$ which is a smooth function and satisfies (1). When the system is periodic in time so that F is explicitly periodic in t, it is advantageous to construct a Poincaré map[1] so that the governing equation for the system takes the form of a point mapping (or a discrete dynamical system)

$$x(n+1) = G(x(n)), \qquad x \in R^N, \quad n \in Z \tag{2}$$

Along the flow of (1) the state space volume changes locally according to $\nabla \cdot F$. For dissipative systems the state space volume will contract on the average along the flow. This implies that as $t \to \infty$, the initial state volume shrinks to zero and the motion of the system takes place on a set of measure zero, which is called an attractor. We will call a set $A \in U$ an attractor if it is an indecomposible closed invariant set with a neighborhood N such that for all $x \in N$, the ω limit of x is contained in A.[1] There are essentially four kinds of attractors for the dissipative dynamical system (1): fixed points, limit cycles, tori, and strange attractors. They represent the stationary, periodic, quasiperiodic, and chaotic behavior, respectively.

For the point mapping (2), the local state space volume changes by a factor $|\det DG(x)|$ on each iteration, where $DG(x)$ is the Jacobian matrix for G evaluated at x. Again, the long-term behavior of a dissipative system is represented by its attractors.

The attractors of the system (1) will appear as attractors in the point mapping (2). The representation of the attractors of continuous dynamical systems in the point mapping, however, depends on how the point mapping system is constructed. When system (2) is a Poincaré map, the limit cycles appear as periodic points and the tori as limit cycles, while strange attractors remain as strange attractors in the point mapping.

3. ATTRACTORS AS PERSISTENT GROUPS IN THE GCM

Here, we give a very brief discussion of the GCM. For more details and terminology, the reader is referred to Refs. 5 and 6. From the point mapping system (2) the cell mapping is constructed. First the domain of interest in the state space is discretized into a collection of cells with sides h_i. This collection of cells is called the cell state space and is denoted by S. The cells in S are labelled 1, 2,..., $N(S)$ according to an appropriate scheme, where $N(S)$ is the total number of cells in S. The state space points in a cell are represented by a certain subset of them. Under the point mapping (2), this representation subset (or sampling set) of the cell i is mapped into their image points, and the cells to which these image points belong become the image cells of cell i. By counting how many points in the representation subset map into a particular image cell j we assign the probability p_{ji} of cell i being mapped into cell j in one mapping step. If p_{ji} is nonzero, we say that cell j is an image cell of cell i or cell i is a pre-image cell of cell j. The matrix $P=\{p_{ij}\}$, $i, j=1,..., N(S)$, is called the transition probability matrix. The cell probability vector $\zeta(n)$ has components $\zeta_i(n)$, $i=1, 2,..., N(S)$, which denotes the probability of the state of the system being in cell i at the nth step. Now the GCM is described by the following evolution equation in terms of the cell probability vector:

$$\zeta(n+1)=P\zeta(n) \tag{3}$$

It is easy to see that P completely controls the evolution process.

The next step is to classify the cells according to their properties. The n-step transition probability $p_{ij}^{(n)}$ is the probability of being in cell i after n steps starting from cell j and is the (i, j)th element of P^n. Cell j leads to cell i if and only if there exists a positive integer m such that $p_{ij}^{(m)}>0$. The cells i and j communicate if and only if cell i leads to cell j and vice versa. The property of communicativeness is a class property, so that it divides the cells into disjoint subsets. A cell that communicates with every cell to which it leads is called essential. All the essential cells form isolated groups which have the property that the cells in the same group communicate, but cells belonging to different groups do not. These isolated groups are called persistent groups. A cell i for which $p_{ii}=1$ forms a persistent group by itself, and is called an absorbing cell. A persistent group with more than one cell is called either an acyclic group or a periodic group according to whether the period d of the group is one or greater than one. For each persistent group the limiting probability distribution p is obtained, which will be an approximation to the invariant distribution for the attractor the persistent group represents.

From its definition an attractor is an indecomposible closed invariant subset and has a neighborhood it attracts. A persistent group in the GCM is indecomposible since it is composed with essential cells. By considering the transition probability matrix in its normal form,[5] it is clear that the persistent group is isolated and invariant under the evolution law, and it attracts probabilities from the transient groups. Therefore attractors are represented as persistent groups in the GCM. From here on persistent groups will mean the persistent groups with more than one cell unless otherwise noted.

It is easy to see that a stable fixed point is represented as either an absorbing cell or an acyclic persistent group. Consequently, stable periodic points of period k are represented as either absorbing cells or acyclic persistent groups in the GCM constructed from G^k. In the remainder of this section we discuss attractors other than the fixed points and periodic points. The cell size h_i denotes the resolution with which the states of the system in x_i coordinates are measured. Therefore, once the h_i are chosen, there exists a limit on the attractors that can be distinguished. For example, the limit cycle right after the Hopf bifurcation[11] can be arbitrarily small, so that in the world of finite precision it cannot be distinguished from a fixed point. When the cell size is adequately small, the attractors will be represented as persistent groups.

4. THE LYAPUNOV EXPONENTS

Lyapunov exponents are quantitative measures of average exponential divergence or convergence of nearby trajectories of a system.[2] Since they can be computed either for a model or from experimental data, they are widely used for the classification of the attractors. For example, the strange attractors have the property that nearby trajectories have exponential separation locally while confined in a compact subset of the state space globally, thereby yielding at least one positive Lyapunov exponent.

The computation of Lyapunov exponents involves averaging the tangent map along the trajectory. The multiplicative ergodic theorem[12] provides the characterization of the matrix product of the tangent map. We define $DG_n = DG(x_n) \cdots DG(x_1)$, where $DG(x)$ is the Jacobian matrix for the point mapping (2) evaluated at x, and $x_{i+1} = G(x_i)$. Let $(DG_n)^*$ denote the adjoint of DG_n. For almost all x,

$$\lim_{n \to \infty} \{[DG_n(x)]^* DG_n(x)\}^{1/2n} = \Lambda_x \tag{4}$$

exists and the logarithms of its eigenvalues are called Lyapunov characteristic exponents, denoted by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$.[13]

The largest (or maximal) Lyapunov exponent λ characterizes the type of attractor in dynamical systems; the fixed point has $\lambda < 0$ and the limit cycles $\lambda = 0$, while strange attractors have $\lambda > 0$. In this paper we will be concerned only with the largest Lyapunov exponent. When the largest Lyapunov exponent has multiplicity one, there exists a field of unit vectors $w(x)$ such that

$$DG(x_i)\, w(x_i) = a(x_i)\, w(x_{i+1}) \tag{5}$$

For the numerical computation of the largest Lyapunov exponent, direct application of the definition (4) is not satisfactory because of the repeated multiplication of the Jacobian matrix evaluated along the trajectory. Therefore, Benettin *et al.*[14] have proposed a scheme which utilizes the linearity and the composition law of the tangent map,[15] since the Jacobian matrix is the tangent map for the vectors. The scheme keeps track of a vector as it evolves under the tangent map. When the largest Lyapunov exponent is positive, the length of the vector increases exponentially, causing overflow problems. To evercome this difficulty of overflow in the computation, the vector is renormalized after a certain number of iterations.

We briefly outline the scheme with renormalization at each iteration. Choose an initial condition x_0 in the domain of attraction $U \subset R^N$ for the attractor in question. Let E_n be the tangent space at point $x_n [= G^n(x_0)]$, i.e., $E_n = T_{x_n} R^N$, and in this case $E_n \equiv R^N$. Choose a unit vector $w_0 \in E_0$ at random. The recursive relation is then defined to be

$$\begin{aligned} \alpha_k &= |DG(x_{k-1})\, w_{k-1}| \\ w_k &= \pm \frac{DG(x_{k-1})\, w_{k-1}}{\alpha_k}, \qquad k \geqslant 1 \end{aligned} \tag{6}$$

where $|\cdot|$ denotes the length of a vector. For sufficiently large k, w_k can be expected to approach either $+w(x_k)$ or $-w(x_k)$ of (5). The largest Lyapunov exponent is given by

$$\lambda = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \ln \alpha_i \tag{7}$$

Equation (5) implies that $DG(x_i)\, w(x_i)$ and $w(x_{i+1})$ are two vectors which may be in the same direction or in the opposite direction. In the former case $a(x_i)$ is positive, while in the latter case it is negative. In the algorithm computing the largest Lyapunov exponent, the crucial quantity is α_k. Therefore, we can use either w_{k-1} or $-w_{k-1}$ in the first equation of (6), leading to the same result on α_k. This, in turn, allows us to restrict the w_i, $i = 1, 2, \ldots$, vectors to a set of vectors satisfying the condition $w_i \cdot e > 0$, where e is a conveniently chosen unit vector.

The proposed GCM algorithm for computing the largest Lyapunov exponent is mostly an implementation of the above scheme in the framework of GCM. Consider a motion representing the strange attractor of (2) under study. Let the trajectory of this motion be given by x_i, $i = 1, 2, \ldots$. Starting with an arbitrary unit vector w_0 in the tangent space at x_0, the tangent map described above and the normalization procedure yield a sequence of unit vectors w_i, $i = 1, 2, \ldots$, with w_i in the tangent space at x_i.

Now consider a persistent group B which represents the strange attractor in the GCM method. Let $N(B)$ be the number of cells in B. If the cells are sufficiently small and if they do not contain a periodic point, then (7) implies that all the w_l associated with x_l that are located in one cell, say cell j, will be nearly equal if l is sufficiently large, and may all be represented with sufficient accuracy by an appropriately defined average unit *flow vector* $u(j)$ associated with the cell j. This flow vector $u(j)$ for cell j is the approximation to the field of unit vectors defined in (5) at the points inside the cell j. We then compute the Jacobian of the point mapping $DG(x_{(j)})$ at the center point $x_{(j)}$ of cell j and evaluate

$$\alpha(j) = |DG(x_{(j)})\, u(j)| \tag{8}$$

which is the cell counterpart of α_k given in (6), and which also approximates the value $|a(x_i)|$ in (5) at the cell j. The largest Lyapunov exponent is now computed by

$$\lambda = \sum_{j=1}^{N(B)} p_j \ln \alpha(j) \tag{9}$$

where p_j is the limiting probability of cell j.

Next, we still need to investigate how $u(j)$, $j=1, 2,..., N(B)$ for various cells are related to each other. Consider the $N(B)$ cells of the persistent group. At each cell j there is a unit cell flow vector $u(j)$. The tangent mapping $DG(x_{(j)})$ of $u(j)$ yields a vector $DG(x_{(j)})\, u(j)$ which is to be assigned to all the image cells of cell j. Consider now a cell k. Suppose that the pre-image cells of cell k are cells $j_1, j_2,..., j_m$. Then $u(k)$ will be related to the vectorial sum of the contributions of tangent mappings of $u(j_1)$, $u(j_2),..., u(j_m)$ from cells $j_1, j_2,..., j_m$. These contributions should, however, be weighted by the limiting probabilities $p_{j_1}, p_{j_2},..., p_{j_m}$ of the pre-image cells and also the transition probabilities $p_{kj_1}, p_{kj_2},..., p_{kj_m}$. Thus,

$$u(k)=\frac{\sum_{i=1}^{m}\{\pm DG(x_{(j_i)})\, u(j_i)\}\, p_{j_i} p_{kj_i}}{|\sum_{i=1}^{m}\{\pm DG(x_{(j_i)})\, u(j_i)\}\, p_{j_i} p_{kj_i}|}$$

$$=\frac{\sum_{j=1}^{N(B)}\{\pm DG(x_{(j)})\, u(j)\}\, p_j p_{kj}}{|\sum_{j=1}^{N(B)}\{\pm DG(x_{(j)})\, u(j)\}\, p_j p_{kj}|} \tag{10}$$

Here the presence of $\pm$ signs is based upon the reasoning given in the paragraph following (7). This equation relates $u(j)$ vectors to each other. In the algorithm it will be used as an updating formula for $u(j)$ from one iterative cycle to the next. At the nth cycle the set of unit flow vectors will be denoted by $u_n(j)$.

Equations (8)–(10) are the bases for the proposed algorithm, which is an iterative one. The algorithm begins with a set of initiating steps.

(i) An arbitrary initial unit vector u_0 is assigned to all cells of the persistent group, i.e., $u_0(j)=u_0$, $j=1, 2,..., N(B)$.

(ii) Compute $\alpha_0(j)$ from (8) and compute λ_0 from (9). Here the subscript 0 has been appended to α and λ to indicate they are for the 0th iterative cycle.

Next, we begin the iterative cycles. A typical nth cycle, $n=1, 2,...$, consists of the following steps.

1. Using $u_{n-1}(j)$, compute a set of updated $u_n(j)$ by (10).
2. Use $u_n(j)$ for $u(j)$ on the right-hand side of (8) to compute $\alpha_n(j)$, $\alpha(j)$ for the nth cycle.
3. Use $\alpha_n(j)$ for $\alpha(j)$ on the right-hand side of (9) to compute λ_n, λ for the nth cycle.
4. Let the Cesàro sum of λ_n be $\lambda(n)$,

$$\lambda(n)=\frac{1}{n}\sum_{j=1}^{n}\lambda_j \tag{11}$$

5. If $|\lambda(n)-\lambda(n-1)|<\delta$ for a predetermined δ, then the Cesàro sum $\lambda(n)$ is considered to have converged and is taken to be the largest Lyapunov exponent. If the Cesàro sum $\lambda(n)$ has not converged, then repeat the iterative steps 1–5. Here $\lambda(0)$ will be assumed to be equal to λ_0.

In the cases of strange attractors and limit cycles examined in this paper, the vectors $u_n(j)$ also converge, which in turn makes λ_n converge as well as its Cesàro sum $\lambda(n)$. For a persistent group that represents a stable spiral point, the vectors $u_n(j)$ at certain cells rotate from step to step and therefore do not converge. However, the Cesàro sum $\lambda(n)$ does converge.

In summary, this scheme determines a flow vector for each cell, and computes the largest Lyapunov exponent in a spatial averaging process, rather than by temporal averaging as for other methods.

5. EXAMPLES

To test the proposed algorithm, we apply it to several well-known examples. First consider the stretch-contraction-reposition map[7,16]

$$\begin{aligned} x_2(n+1) &= \lambda_1 x_2(n) \bmod 1 \\ x_1(n+1) &= \lambda_2 x_1(n) + x_2(n) - \frac{1}{\lambda_1} x_2(n+1) \end{aligned} \tag{12}$$

where λ_1 is a positive integer greater than 1 and λ_2 is a positive number. The Jacobian matrix is constant for all x,

$$DG(x) = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} \tag{13}$$

Therefore all the cells in the persistent group B have

$$u(i) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \alpha(i) = \lambda_1, \qquad i = 1, 2, ..., N(B) \tag{14}$$

so that the largest Lyapunov exponent is $\ln \lambda_1 > 0$, indicating that the persistent group represents the strange attractor of the system (12).

A second example is the nonlinear point mapping[6]

$$\begin{aligned} x_1(n+1) &= 0.9x_2(n) + 1.81x_1^2(n) \\ x_2(n+1) &= -0.9x_1(n) \end{aligned} \tag{15}$$

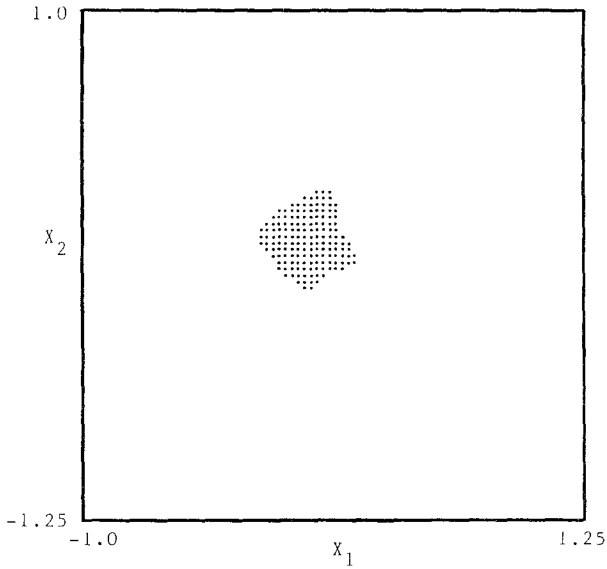

Fig. 1. The persistent group for the nonlinear point mapping system (15).

The GCM is constructed from the state space $-1.0 \leqslant x_1 < 1.25$, $-1.25 \leqslant x_2 < 1.0$ with $h_1 = h_2 = 0.028125$. The sampling is 7×7. There is an acyclic persistent group of 156 cells near the origin as shown in Fig. 1; see also Figs. 4–6 in Ref. 6. Most of the cells have a limiting probability distribution of less than 10^{-6}. Table I shows the results of the computation for the largest Lyapunov exponent on the VAX 11/750. In all the computations in this section the first several values of the λ_j were ignored in Eq. (11) to expedite the convergence and δ was chosen to be 0.0001. Usually convergence of $\lambda(n)$ is reached quite rapidly. For this case, where the persistent group represents a stable spiral fixed point, the flow vector of the cell to which the fixed point belongs rotates approximately 90 deg at

Table I. The Largest Lyapunov Exponents

System equation	GCM	Benettin's scheme
(15)	−0.105	−0.105
(16)	0.002	0.0001
(17)	0.42	0.42
(18)	0.66	0.64

each iteration, reflecting the nature of the fixed point. This effect prevents convergence of flow vectors in this case. The Cesàro sum $\lambda(n)$ does, however, converge. The negative Lyapunov exponent indicates that the persistent group corresponds to the stable fixed point at the origin.

For an example of the limit cycle, we choose the Van der Pol equation

$$\ddot{x} + (1 - x^2)\,\dot{x} + x = 0 \tag{16}$$

For this system we integrate the equation for a time interval 1 and the cell mapping is set up for $-3.0 \leqslant x < 3.0$, $-3.0 \leqslant \dot{x} < 3.0$ with $h_1 = h_2 = 0.2$. The sampling was 5×5, the same as the rest of the examples in this section. When the Jacobian matrix is not explicitly available as in this case, it has to be determined numerically. In most methods the approximate Jacobian matrix has to be determined by computing additional point mapping for the points near the trajectory, requiring more numerical integrations. But for the GCM additional integrations are not necessary, since they are already done in the construction of the transition probability matrix.

Applying the GCM to (16), one finds that there is a persistent group corresponding to the limit cycle indicated with a Lyapunov number close to zero. For this case, as well as the next two examples, both the flow vectors $u_n(j)$ at each cell and λ_n converge. The flow vectors for the persistent group can be seen approximately in the flow direction of the limit cycle; see Fig. 2. The slight deviation of the Lyapunov exponent from zero is

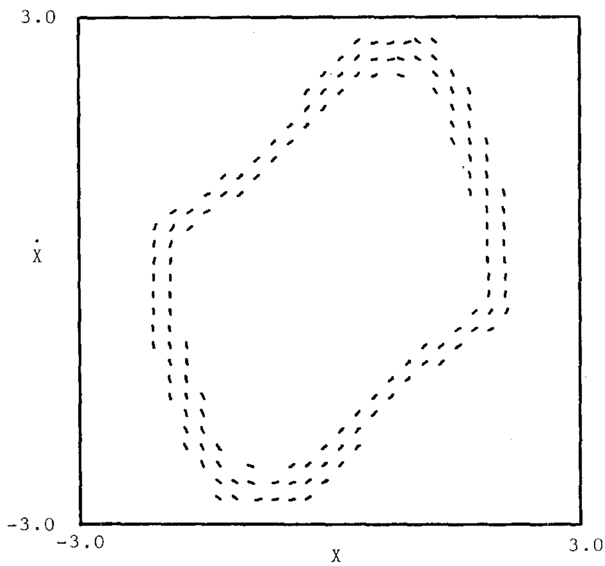

Fig. 2. The flow vectors of the persistent group for the Van der Pol equation (16).

reasonable, because the persistent group covers the limit cycle with a finite width across it.

The Hénon–Pomeau map[17]

$$x_1(n+1) = 1 + x_2(n) - a x_1^2(n)$$
$$x_2(n+1) = b x_1(n) \tag{17}$$

has a well-known strange attractor for the parameters $a = 1.4$, $b = 0.3$. The region in the state space $-1.5 \leqslant x_1 < 1.5$, $-0.5 \leqslant x_2 < 0.5$ is set up into a cell space of 900×900 cells. The data in the Table I indicate that the persistent group shown in Fig. 2 of Ref. 7 corresponds to a strange attractor. The flow vectors for cells near the unstable fixed point are shown in Fig. 3.

Finally, we consider the Duffing equation in the form[18]

$$\ddot{x} + k\dot{x} + x^3 = B \cos t \tag{18}$$

with $k = 0.05$, $B = 7.5$. The cell space is constructed with 100×100 cells covering $1 \leqslant x < 4$, $-6 \leqslant \dot{x} < 6$. A Poincaré map for the differential equation (18) is constructed by numerically integrating the equation over one period. The persistent group representing the strange attractor is shown in Fig. 4a. The flow vectors for a part of the persistent group cover-

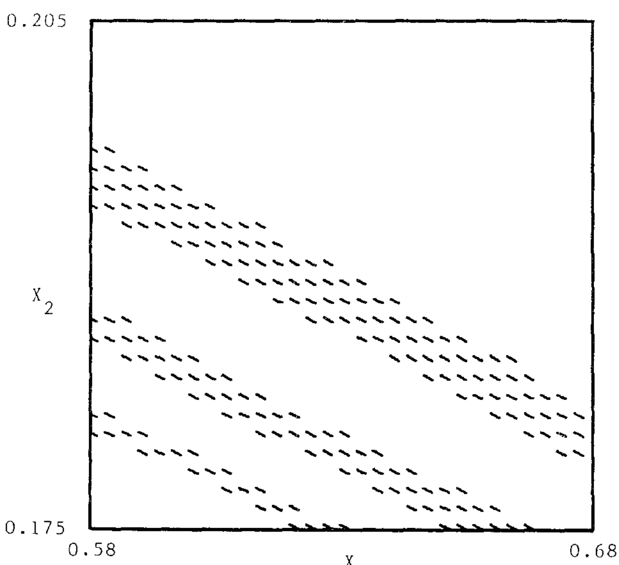

Fig. 3. The flow vectors for cells near the unstable fixed point of the Hénon–Pomeau map (17).

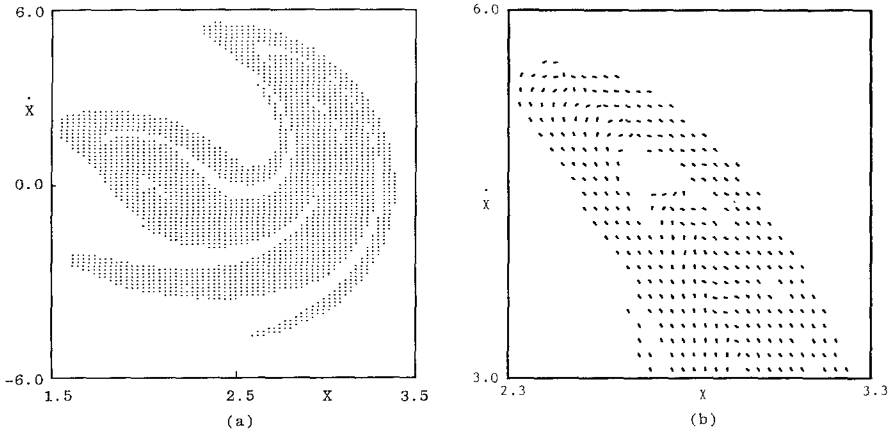

Fig. 4. (a) The persistent group for the Duffing equation (18), and (b) the flow vectors for a part of the persistent group in (a).

ing $2.3 < x < 3.3$, $3.0 < \dot{x} < 6.0$ are shown in Fig. 4b. The large, positive Lyapunov exponent of value 0.66 indicates that the persistent group corresponds to a strange attractor. It is to be noted that the exponent 0.66 reported here is the largest Lyapunov exponent for the corresponding Poincaré map of the Duffing system. This value should be divided by 2π, the period of excitation for (18), to yield the largest Lyapunov exponent for the flow. The authors are grateful to Professor F. Moon for pointing out the need to clarify this possible point of confusion.

6. CONCLUDING REMARKS

In this paper we have presented an algorithm for the largest Lyapunov exponent by the generalized cell mapping method. The data obtained for several examples agree very well with the Lyapunov exponents obtained with Benettin's scheme. This has opened a way to characterize the persistent groups and absorbing cells by generalized cell mapping.

We also remark here that in using GCM to study a nonlinear system we will usually find persistent groups. But these persistent groups could represent equilibrium states, limit cycles, quasiperiodic solutions, and/or strange attractors. We need a method which will allow us to differentiate these cases. The proposed algorithm in this paper can serve that purpose; therefore, it will become an integral part of the methodology of GCM.

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